Two-State Markovian Representations of Term Structure Dynamics *

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Abstract

This paper considers a family of models for pricing interest rate derivatives where the term structure is Markov with respect to two state variables. These models belong to the class of Heath, Jarrow, and Morton models in that the initial forward rate curve and volatility structures are inputs to the model. In this family, the forward rate volatility structure at time t is a function of time, maturity date, the spot interest rate at time t, and a state variable which captures the history of the Brownian motion up to time t.

1 Introduction

Interest rate derivatives are financial instruments whose payoffs depend on the level of interest rates. These instruments have become increasingly popular in recent years, and there has been considerable interest in the valuation of these derivatives. There are a number of models used to value interest-rate-contingent claims, including Vasicek (1977), Cox, Ingersoll and Ross (1985), Black, Derman and Toy (1990), and Heath, Jarrow and Morton (1992). (Hereafter CIR, BDT, and HJM) Each of these models has its own advantages and limitations. The term structure is an output of Vasicek and CIR models, so they cannot fit the initial yield curve. The BDT model is widely used among practitioners, because the initial forward rate curve and initial volatility structure can be fitted. Another advantage is that the interest rate process generated by BDT is Markov, so recombining lattices can be used to value interest rate options.

The HJM model is the most general, and it contains all of the aforementioned models, as well many others. Unfortunately the interest rate process for the HJM paradigm is, in general, non-Markov, so recombining lattices cannot be used. An alternative, Monte Carlo simulation, which can be used to implement this model of the term structure, is computationally expensive, and cannot be used to value American-style options. In summary, the existing Markovian Models are not general enough, and the HJM framework is difficult to implement.

This paper overcomes these difficulties by considering a class of models under the HJM framework which are Markovian with respect to two state variables. Hull and White (1993a) and Carverhill (1994) find the form that deterministic forward rate volatilities must have if the spot rate follows a Markov process. (Deterministic forward rate volatilities are functions of time, t, and maturity date, T, only.) Jeffery (1995) finds conditions that forward rate volatility functions of r, t, and T must satisfy for the spot rate process to be Markov with respect to one state variable, the interest rate, r. Ritchken and Sankarasubramanian (hereafter RS) (1995) examine a class of models which are Markovian with respect to two state variables,

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r and ϕ, the interest rate and an integrated variance factor of form $\int_0^t \sigma^f(r,\tau,t)^2 d\tau$ where $\sigma^f(r,\tau,T)$ is the forward rate volatility. However, this class of models is incomplete, since Jeffery considers models which are not contained in RS, but which are Markovian with respect to one state variable, as well as two state variables.

This paper attempts to find the complete set of two-state Markovian models, by considering conditions on the forward rate volatilities that must be satisfied in order for the spot rate process to be Markov with respect to two state variables. The approach to the problem in this paper is in essentially the same manner as in Jeffery (1995).

2 The Heath-Jarrow-Morton Framework

The HJM framework models the evolution of the instantaneous forward rate, $f(t,T)$, which is defined by

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T)$$

where $P(t,T)$ denotes the price at time t of a bond which has a payoff of one dollar at T, the time of maturity.

Forward rates in a one-factor HJM framework are assumed to follow a stochastic differential equation of the form

$$df(t,T) = \mu^f(\omega,t,T)dt + \sigma^f(\omega,t,T)dz(t)$$

where $z(t)$ is standard Brownian motion, and ω represents the history of the Brownian motion. If there is no arbitrage, then under an equivalent risk-neutral martingale measure, HJM show that

$$\mu^f(\omega,t,T) = \sigma^f(\omega,t,T)\int_t^T \sigma^f(\omega,t,u)du$$

It follows that the evolution of the forward rates is given by the SDE

$$df(t,T) = \sigma^f(\omega,t,T)\int_t^T \sigma^f(\omega,t,u)du\,dt + \sigma^f(\omega,t,T)dz(t) \tag{1}$$

The short rate process can be derived under the risk-neutral measure

$$\begin{aligned} r(t) &= f(t,t) \\ &= f(0,t) + \int_0^t df(\tau,t) \\ &= f(0,t) + \int_0^t \mu^f(\omega,\tau,t)d\tau + \int_0^t \sigma^f(\omega,\tau,t)dz(t) \\ &= f(0,t) + \int_0^t \sigma^f(\omega,\tau,t)\int_\tau^t \sigma^f(\omega,\tau,u)du\,d\tau + \int_0^t \sigma^f(\omega,\tau,t)dz(t) \end{aligned}$$

It follows that

$$\begin{aligned} dr(t) = f_t(0,t) &+ \left\{\int_0^t [\sigma_t^f(\omega,\tau,t)\int_\tau^t \sigma^f(\omega,\tau,u)du + \sigma^f(\omega,\tau,t)^2]d\tau + \int_0^t \sigma_t^f(\omega,\tau,t)dz(\tau)\right\}dt \\ &+ \sigma^f(\omega,t,t)dz(t) \end{aligned}$$

As Hull and White (1993b) point out, the second term depends on the history of the Brownian motion. So, in general, the process for $r(t)$ will not be Markov.

3 An Example of a Two-State Markovian Paradigm

The spot interest rate dynamics are given by

$$dr = \mu^r(r,\phi,t)dt + \sigma^r(r,t)dz(t) \tag{2}$$

$$d\phi = \mu^\phi(r,\phi,t)dt \tag{3}$$

We will consider models whose forward rates are smooth functions of r, ϕ, t, and T, and whose forward rate volatilities are smooth functions of r, t, and T. In the general case, which is considered in Section 5, the process for ϕ has a diffusion term. However, to simplify the exposition, ϕ is modeled without a diffusion term in this section. Even with this restriction, many volatility structures are allowable, including those considered by RS (1995).

Applying Itô's lemma to $f(r,\phi,t,T)$ yields

$$\begin{aligned} df(r,\phi,t,T) = & \left\{ \frac{\partial f(r,\phi,t,T)}{\partial r}\mu^r(r,\phi,t) + \frac{\partial f(r,\phi,t,T)}{\partial \phi}\mu^\phi(r,\phi,t) + \frac{\partial f(r,\phi,t,T)}{\partial t} \right. \\ & \left. + \frac{1}{2}\frac{\partial^2 f(r,\phi,t,T)}{\partial r^2}\sigma^r(r,t)^2 \right\} dt \\ & + \frac{\partial f(r,\phi,t,T)}{\partial r}\sigma^r(r,t)dz(t) \end{aligned} \tag{4}$$

Equating the drift and diffusion terms of (1) and (4), yields the following equations

$$\begin{aligned} \sigma^f(r,t,T)\int_t^T \sigma^f(r,t,u)du = & \frac{\partial f(r,\phi,t,T)}{\partial r}\mu^r(r,\phi,t) + \frac{\partial f(r,\phi,t,T)}{\partial \phi}\mu^\phi(r,\phi,t) + \frac{\partial f(r,\phi,t,T)}{\partial t} \\ & + \frac{1}{2}\frac{\partial^2 f(r,\phi,t,T)}{\partial r^2}\sigma^r(r,t)^2 \end{aligned} \tag{5}$$

$$\sigma^f(r,t,T) = \frac{\partial f(r,\phi,t,T)}{\partial r}\sigma^r(r,t) \tag{6}$$

with the boundary conditions

$$f(r,\phi,t,t) = r \tag{7}$$

$$\sigma^f(r,t,t) = \sigma^r(r,t) \tag{8}$$

From (6) and (8)

$$\frac{\partial f(r,\phi,t,T)}{\partial r} = \frac{\sigma^f(r,t,T)}{\sigma^f(r,t,t)} \tag{9}$$

Differentating both sides by r yields

$$\frac{\partial^2 f(r,\phi,t,T)}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{\sigma^f(r,t,T)}{\sigma^f(r,t,t)}\right] \tag{10}$$

Integrating by r and using (9)

$$f(r,\phi,t,T) = \int_0^r \frac{\sigma^f(m,t,T)}{\sigma^f(m,t,t)}dm + g(\phi,t,T) \tag{11}$$

Then differentiating by t and ϕ

$$\frac{\partial f(r,\phi,t,T)}{\partial t} = \frac{\partial}{\partial t}\left[\int_0^r \frac{\sigma^f(m,t,T)}{\sigma^f(m,t,t)}dm\right] + \frac{\partial}{\partial t}g(\phi,t,T) \tag{12}$$

$$\frac{\partial f(r,\phi,t,T)}{\partial \phi} = \frac{\partial g(\phi,t,T)}{\partial \phi} \tag{13}$$

Theorem 3.1 *A volatility structure $\sigma^f(r,t,T)$ is allowable in a two-state Markovian paradigm given by (2-3) if and only if there exist functions $g(t,T,\phi)$, $\mu^\phi(r,\phi,t)$ and $\mu^r(r,\phi,t)$ which satisfy*

$$\begin{aligned}\sigma^f(r,t,T)\int_t^T \sigma^f(r,t,u)du &= \frac{\sigma^f(r,t,T)}{\sigma^f(r,t,t)}\mu^r(r,\phi,t) + \frac{\partial g(\phi,t,T)}{\partial\phi}\mu^\phi(r,\phi,t) \\ &\quad + \frac{\partial}{\partial t}\left[\int_0^r \frac{\sigma^f(m,t,T)}{\sigma^f(m,t,t)}dm\right] + \frac{\partial}{\partial t}g(\phi,t,T) + \frac{1}{2}\sigma^f(r,t,t)^2\frac{\partial}{\partial r}\left[\frac{\sigma^f(r,t,T)}{\sigma^f(r,t,t)}\right] \qquad (14)\\ \mu^r(r,\phi,t) &= -\frac{\partial}{\partial t}\left[\int_0^r \frac{\sigma^f(m,t,T)}{\sigma^f(m,t,t)}dm\right]\Bigg|_{T=t} - \frac{\partial g(\phi,t,T)}{\partial t}\Bigg|_{T=t} \qquad (15)\end{aligned}$$

Proof: Suppose that $\sigma^f(r,t,T)$ is allowable in a two-state Markovian paradigm. Then since the interest dynamics are given by (2-3), (4-13) hold. To obtain (14), substitute (9-13) into (5). To obtain (15), evaluate (14) at $T=t$.

Now suppose that (14-15) hold for some $\sigma^f(r,t,T)$, $g(t,T,\phi)$, $\mu^\phi(r,\phi,t)$ and $\mu^r(r,\phi,t)$. Then define a two-state Markovian interest rate process by $dr = \mu^r(r,\phi,t)dt + \sigma^f(r,t,t)dz(t)$ and $d\phi = \mu^\phi(r,\phi,t)dt$, with a corresponding forward rate volatility $\sigma^f(r,t,T)$. By (4-13), (14-15) can be obtained with the *same* $\sigma^f(r,t,T)$ that was orginally considered. Therefore the volatility stucture is indeed given by a two-state Markovian process. □

This result can be restated as

$$\begin{aligned}&-\frac{\sigma^f(r,t,T)}{\sigma^f(r,t,t)}g_t(\phi,t,t) + \frac{\partial}{\partial\phi}\left[\int_t^T g_t(\phi,\tau,T)d\tau\right]\mu^\phi(r,\phi,t) + g_t(\phi,t,T) = \sigma^f(r,t,T)\int_t^T \sigma^f(r,t,u)du \\ &+\frac{\sigma^f(r,t,T)}{\sigma^f(r,t,t)}\frac{\partial}{\partial t}\left[\int_0^r \frac{\sigma^f(m,t,T)}{\sigma^f(m,t,t)}dm\right]\Bigg|_{T=t} - \frac{\partial}{\partial t}\left[\int_0^r \frac{\sigma^f(m,t,T)}{\sigma^f(m,t,t)}dm\right] - \frac{1}{2}\sigma^f(r,t,t)^2\frac{\partial}{\partial r}\left[\frac{\sigma^f(r,t,T)}{\sigma^f(r,t,t)}\right] \qquad (16)\end{aligned}$$

Let $\tilde{g}_t(\phi,t,T)$ be an arbitrary $g_t(\phi,t,T)$ which satisfies (16). Then, since the right hand side does not depend on the choice of $g_t(\phi,t,T)$, the allowable forms of $g_t(\phi,t,T)$ are solutions of

$$\begin{aligned}g_t(\phi,t,T) &= \tilde{g}_t(\phi,t,T) + \frac{\sigma^f(r,t,T)}{\sigma^f(r,t,t)}\left(g_t(\phi,t,t) - \tilde{g}_t(\phi,t,t)\right) \\ &\quad -\frac{\partial}{\partial\phi}\left[\int_t^T \left(g_t(\phi,\tau,T) - \tilde{g}_t(\phi,\tau,T)\right)d\tau\right]\mu^\phi(r,\phi,t) \qquad (17)\end{aligned}$$

Now, suppose there is a $\tilde{g}_t(\phi,t,T)$ which satisfies (16). Then the initial forward rate curve is given by (11) with $t=0$

$$f(r,\phi,0,T) = \int_0^r \frac{\sigma^f(m,0,T)}{\sigma^f(m,0,0)}dm + \tilde{g}(\phi,0,T) = \tilde{F}(T)$$

where $\tilde{F}(T)$ is an initial forward rate curve at $t=0$, $r=r(0)$, and $\phi=\phi(0)$ determined by $\tilde{g}(\phi,0,T)$. However, in general, $\tilde{g}(\phi,0,T)$ will not match an arbitrary $F(T)$, so it is necessary to find a $g(\phi,t,T)$ that will fit an arbitrary initial forward rate curve.

If $\sigma^f(r,t,T)$ is of the form $\xi(t,T)\sigma^r(r,t)$, a deterministic function of time multiplied by the volatility of the spot interest rate, then it will be shown that an appropriate $g(\phi,t,T)$ can be chosen.

Theorem 3.2 *Suppose that $\sigma^f(r,t,T)$ is a volatility structure of the form $\xi(t,T)\sigma^r(r,t)$ which satisfies (16). Fix $r=r(0)$ and $\phi=\phi(0)$. Then, for any differentiable function $k(t)$, where $k(0)=0$, there exists a function $g(\phi,t,T)$ satisfying (16) and such that $g(\phi,0,T)=k(T)$. In particular, $k(T)$ can be chosen to be $F(T) - \int_0^r \frac{\sigma^f(m,0,T)}{\sigma^f(m,0,0)}dm$ so that*

$$\int_0^r \frac{\sigma^f(m,0,T)}{\sigma^f(m,0,0)}dm + g(\phi,0,T) = F(T)$$

Proof: Since $f(r,t,t) = r$, it follows from equation (11) that $g(\phi,t,t) = 0$. Therefore, $g(\phi,t,T) = \int_t^T g_t(\phi,\tau,T)d\tau$. So, k(T) is given by

$$k(T) = g(\phi,0,T) = \int_0^T g_t(\phi,\tau,T)d\tau$$

To prove the theorem, it suffices to show the existence of a $g_t(\phi,t,T)$ which satisfies equation (17) and the above equation for arbitrary differentiable function $k(t)$. It is claimed that $g_t(\phi,t,T)$ can be chosen so that $g_t(\phi,t,T) - \tilde{g}_t(\phi,t,T)$ is a function of t and T only. Under this restriction, equation (17) can be rewritten as

$$g_t(\phi,t,T) = \tilde{g}_t(\phi,t,T) + \xi(t,T)\left(g_t(\phi,t,t) - \tilde{g}_t(\phi,t,t)\right)$$

Fix ϕ, $\tilde{g}_t(\phi,t,t)$, and $\xi(t,T)$. It needs to be shown that there is a $g(\phi,t,t)$ such that

$$k(T) = \int_0^T \left[\tilde{g}_t(\phi,\tau,T) + \xi(\tau,T)\left(g_t(\phi,\tau,\tau) - \tilde{g}_t(\phi,\tau,\tau)\right)\right]d\tau$$

Which can be rewritten as

$$k(T) = \int_0^T \tilde{g}_t(\phi,\tau,T)d\tau + \int_0^T \xi(\tau,T)G(s)d\tau$$

Where $G(t) = g_t(\phi,t,t) - \tilde{g}_t(\phi,t,t)$. Since ϕ is fixed, this equation is a Volterra integral equation of the first kind for which there exists a unique solution $G(t)$ for any $K(t)$ with $k(0) = 0$. Therefore $g(\phi,t,t) = G(t) + \tilde{g}_t(\phi,t,t)$ solves the above equations. □

4 A Volatility Structure of the Form $\sigma^f(r,t,T) = \xi(t,T)\sigma^r(r,t)$

Let the volatility structure be of the form $\sigma^f(r,t,T) = \xi(t,T)\sigma^r(r,t)$ Then, from (14-15),

$$\begin{aligned} \sigma^r(r,t)^2\xi(t,T)\int_t^T \xi(t,\tau)d\tau \quad &= \quad \xi(t,T)\left(-r\left.\frac{\partial\xi(t,T)}{\partial t}\right|_{T=t} - \left.\frac{\partial g(\phi,t,T)}{\partial t}\right|_{T=t}\right) \\ &\quad +\frac{\partial g(\phi,t,T)}{\partial\phi}\mu^\phi(r,\phi,t) \\ &\quad +r\frac{\partial\xi(t,T)}{\partial t} + \frac{\partial g(\phi,t,T)}{\partial t} \end{aligned} \tag{18}$$

4.1 Spot interest rate volatilities of the form $\sigma^r(r,t) \neq \sqrt{a(t)r + b(t)}$

If $\sigma^r(r,t)^2$ does not have the form $a(t)r + b(t)$ then by setting equal the terms that are nonlinear in r in (18),

$$\frac{\partial g(\phi,t,T)}{\partial\phi} = d(t)\xi(t,T)\int_t^T \xi(t,\tau)d\tau \tag{19}$$

and

$$\mu^\phi(r,\phi,t) = \frac{1}{d(t)}\sigma^r(r,t)^2 + c_1(t)\phi + c_2(t)r \tag{20}$$

For some function $d(t)$. Now, g can be rewritten as $g(\phi,t,T) = g^\phi(\phi,t,T) + g^t(t,T)$. Therefore,

$$g(\phi,t,T) = \phi d(t)\xi(t,T)\int_t^T \xi(t,\tau)d\tau + g^t(t,T)$$

There are three types of terms in equation (18) in addition to $\sigma^r(r,t)^2\xi(t,T)\int_t^T \xi(t,\tau)d\tau$. They are of the form $\phi f(t,T)$, $rf(t,T)$, and $f(t,T)$. Since the left hand side of equation (18) has no ϕ dependence, the terms of the form $\phi f(t,T)$ must add to zero on the right hand side

$$\xi(t,T)\left.\frac{\partial g^\phi(\phi,t,T)}{\partial t}\right|_{T=t} = \frac{\partial g^\phi(\phi,t,T)}{\partial t} + c_1(t)\phi\frac{\partial g(\phi,t,T)}{\partial \phi}$$

by substituting (19) and dividing by $d(t)$ and ϕ,

$$\xi(t,T)\xi(t,t) = \frac{\partial \xi(t,T)}{\partial t}\int_t^T \xi(t,\tau)d\tau + \xi(t,T)\int_t^T \frac{\partial \xi(t,\tau)}{\partial t}d\tau + \xi(t,T) + \left(c_1(t) + \frac{d'(t)}{d(t)}\right)\xi(t,T)\int_t^T \xi(t,\tau)d\tau$$

Since $\xi(t,t) = 1$,

$$\frac{\partial \xi(t,T)}{\partial t}\int_t^T \xi(t,\tau)d\tau + \xi(t,T)\int_t^T \frac{\partial \xi(t,\tau)}{\partial t}d\tau = \left(c_1(t) + \frac{d'(t)}{d(t)}\right)\xi(t,T)\int_t^T \xi(t,\tau)d\tau \tag{21}$$

Now, the terms of the form $rf(t,T)$ must be zero on the right hand side of (18).

$$\xi(t,T)\left(\left.\frac{\partial \xi(t,T)}{\partial t}\right|_{T=t}\right) = c_2(t)d(t)\xi(t,T)\int_t^T \xi(t,\tau)d\tau + \frac{\partial \xi(t,T)}{\partial t} \tag{22}$$

Finally, the terms of the form $f(t,T)$ must equal zero on the right hand side of (18).

$$\xi(t,T)\left.\frac{\partial g^t(t,T)}{\partial t}\right|_{T=t} = \frac{\partial g^t(t,T)}{\partial t} \tag{23}$$

Since $\xi(t,t) = 1$, an appropriate $g^t(t,T)$ can be chosen to satisfy (23). Therefore all forms of $\xi(t,T)$ satisfy (23). However, $\xi(t,T)$ must also satisfy the other two equations (21-22) for it to be representable as a two-state Markovian process.

Example 4.1 (Ritchken and Sankarasubramanian) We consider

$$\xi(t,T) = e^{\left(-\int_t^T \kappa(\tau)\,d\tau\right)}$$

which can be shown a solution to (21-22). To see this, let $c_1(t) = -2\kappa(t)$, $d(t) = 1$, and $c_2(t) = 0$. The forward rate volatility, $\sigma^f(r,t,T) = \sigma^r(r,t)e^{\left(-\int_t^T \kappa(\tau)\,d\tau\right)}$, is the one examined by RS (1995). It can be shown that this is the complete set solutions of the equations (21-22).

4.2 Spot interest rate volatilities of the form $\sigma^r(r,t) = \sqrt{a(t)r + b(t)}$

Assume that spot rate volatility is of the form $\sigma^r(r,t)^2 = a(t)r + b(t)$, and $g(\phi,t,T)$ is linear in ϕ. Then, since all the other terms in (18) are linear in ϕ and r, let

$$\mu^\phi(r,\phi,t) = c_1(t)\phi + c_2(t)r$$

There are three types of terms in equation (18) which are of the form $\phi f(t,T)$, $rf(t,T)$, and $f(t,T)$. Therefore, the following three equations must be satisfied. Equating the $f(t,T)$ terms on both sides,

$$b(t)\xi(t,T)\int_t^T \xi(t,\tau)d\tau + \xi(t,T)\left.\frac{\partial g^t(t,T)}{\partial t}\right|_{T=t} = \frac{\partial g^t(t,T)}{\partial t} \tag{24}$$

Again, an appropriate $g^t(t,T)$ can be chosen to satisfy (24). Therefore all forms of $\xi(t,T)$ satisfy (24). The remaining equations that $\xi(t,T)$ must satisfy are derived by considering the terms of the type $rf(t,T)$ and $\phi f(t,T)$,

$$a(t)\xi(t,T)\int_t^T \xi(t,\tau)d\tau + \xi(t,T)\left(\left.\frac{\partial \xi(t,T)}{\partial t}\right|_{T=t}\right) = \frac{\partial \xi(t,T)}{\partial t} + c_2(t)g_\phi^\phi(\phi,t,T) \tag{25}$$

and

$$\xi(t,T)\left.\frac{\partial g^\phi(\phi,t,T)}{\partial t}\right|_{T=t} = \frac{\partial g^\phi(\phi,t,T)}{\partial t} + c_1(t)\phi g_\phi^\phi(\phi,t,T) \tag{26}$$

Example 4.2 (Carverhill) We consider

$$\begin{aligned}\xi(t,T) &= e^{\left(-\int_t^T \kappa(\tau)\,d\tau\right)}\\ a(t) &= 0\end{aligned}$$

so that $\sigma^f(r,t,T) = b(t)e^{\left(-\int_t^T \kappa(\tau)\,d\tau\right)}$. This satisfies equations (25-26) with $g(\phi,t,T) = 0$ and is the forward rate volatility considered by Hull and White (1993a) and Carverhill (1994).

Example 4.3 We now consider

$$\begin{aligned}\xi(t,T) &= \cos\left(-\int_t^T \kappa(\tau)\,d\tau\right)\\ a(t) &= 0\end{aligned}$$

so that $\sigma^f(r,t,T) = b(t)\cos\left(-\int_t^T \kappa(\tau)\,d\tau\right)$. This satisfies equations (25-26) with $c_1(t) = 0$, $c_2(t) = \kappa(\tau)$, and $g^\phi(\phi,t,T) = \phi\sin\left(-\int_t^T \kappa(\tau)\,d\tau\right)$.

5 The General Two-State Markovian Paradigm with Diffusion in the State Variable

The spot interest rate dynamics are given by

$$dr = \mu^r(r,\phi,t)dt + \sigma^r(r,t)dz(t) \tag{27}$$
$$d\phi = \mu^\phi(r,\phi,t)dt + \sigma^\phi(r,\phi,t)dz(t) \tag{28}$$

Applying Itô's lemma to $f(r,\phi,t,T)$ yields

$$\begin{aligned}df(r,\phi,t,T) = &\left\{\frac{\partial f(r,\phi,t,T)}{\partial r}\mu^r(r,\phi,t) + \frac{\partial f(r,\phi,t,T)}{\partial \phi}\mu^\phi(r,\phi,t) + \frac{\partial f(r,\phi,t,T)}{\partial t}\right.\\ &\left.+\frac{1}{2}\frac{\partial^2 f(r,\phi,t,T)}{\partial r^2}\sigma^r(r,t)^2 + \frac{1}{2}\frac{\partial^2 f(r,\phi,t,T)}{\partial r\partial\phi}\sigma^r(r,t)\sigma^\phi(r,\phi,t) + \frac{1}{2}\frac{\partial^2 f(r,\phi,t,T)}{\partial \phi^2}\sigma^\phi(r,\phi,t)^2\right\}dt\\ &+\left\{\frac{\partial f(r,\phi,t,T)}{\partial r}\sigma^r(r,t) + \frac{\partial f(r,\phi,t,T)}{\partial \phi}\sigma^\phi(r,\phi,t)\right\}dz(t)\end{aligned} \tag{29}$$

Equating the drift and diffusion terms of (1) and (29), yields the following equations

$$\sigma^f(r,t,T)\int_t^T \sigma^f(r,t,u)du = \frac{\partial f(r,\phi,t,T)}{\partial r}\mu^r(r,\phi,t) + \frac{\partial f(r,\phi,t,T)}{\partial \phi}\mu^\phi(r,\phi,t) + \frac{\partial f(r,\phi,t,T)}{\partial t} \tag{30}$$

$$
\begin{aligned}
& +\frac{1}{2}\frac{\partial^2 f(r,\phi,t,T)}{\partial r^2}\sigma^r(r,t)^2 + \frac{1}{2}\frac{\partial^2 f(r,\phi,t,T)}{\partial r \partial \phi}\sigma^r(r,t)\sigma^\phi(r,\phi,t) \\
& +\frac{1}{2}\frac{\partial^2 f(r,\phi,t,T)}{\partial \phi^2}\sigma^\phi(r,\phi,t)^2 \\
\sigma^f(r,t,T) \quad = \quad & \frac{\partial f(r,\phi,t,T)}{\partial r}\sigma^r(r,t) + \frac{\partial f(r,\phi,t,T)}{\partial \phi}\sigma^\phi(r,\phi,t) \qquad (31)
\end{aligned}
$$

with the usual boundary conditions (7-8). Differentiating both sides of (31) by ϕ,

$$
0 = \frac{\partial^2 f(r,\phi,t,T)}{\partial r \partial \phi}\sigma^r(r,t) + \frac{\partial^2 f(r,\phi,t,T)}{\partial \phi^2}\sigma^\phi(r,\phi,t) + \frac{\partial f(r,\phi,t,T)}{\partial \phi}\frac{\partial \sigma^\phi(r,\phi,t)}{\partial \phi}
$$

This can be rewritten as

$$
\frac{1}{2}\frac{\partial^2 f(r,\phi,t,T)}{\partial r \partial \phi}\sigma^r(r,t)\sigma^\phi(r,\phi,t) + \frac{1}{2}\frac{\partial^2 f(r,\phi,t,T)}{\partial \phi^2}\sigma^\phi(r,\phi,t)^2 = -\frac{1}{2}\frac{\partial f(r,\phi,t,T)}{\partial \phi}\frac{\partial \sigma^\phi(r,\phi,t)}{\partial \phi}\sigma^\phi(r,\phi,t) \quad (32)
$$

In addition, from (31)

$$
\frac{\partial f(r,\phi,t,T)}{\partial \phi} = \frac{\sigma^f(r,t,T) - \frac{\partial f(r,\phi,t,T)}{\partial r}\sigma^r(r,t)}{\sigma^\phi(r,\phi,t)} \qquad (33)
$$

To find the functional form of the forward rate, integrate by r

$$
f(r,\phi,t,T) = \int_0^r \frac{\partial f(m,\phi,t,T)}{\partial m} dm + g(\phi,t,T)
$$

Then differentiating by t and ϕ

$$
\begin{aligned}
\frac{\partial f(r,\phi,t,T)}{\partial t} &= \frac{\partial}{\partial t}\left[\int_0^r \frac{\partial f(m,\phi,t,T)}{\partial m} dm\right] + \frac{\partial g(\phi,t,T)}{\partial t} \qquad (34) \\
\frac{\partial f(r,\phi,t,T)}{\partial \phi} &= \frac{\partial}{\partial \phi}\left[\int_0^r \frac{\partial f(m,\phi,t,T)}{\partial m} dm\right] + \frac{\partial g(\phi,t,T)}{\partial \phi} \qquad (35)
\end{aligned}
$$

The following is an extension of Theorem 3.1 to a general two-state Markovian paradigm.

Theorem 5.1 *A volatility structure $\sigma^f(r,t,T)$ is allowable in the general two-state Markovian paradigm given by (27-28) if there exist functions $f_r(r,\phi,t,T)$, $g(t,T,\phi)$, $\mu^\phi(r,\phi,t)$, $\mu^r(r,\phi,t)$ and $\sigma^\phi(r,\phi,t)$ which satisfy*

$$
\begin{aligned}
\sigma^f(r,t,T)\int_t^T \sigma^f(r,t,u)du &= f_r(r,\phi,t,T)\mu^r(r,\phi,t) + \frac{\partial}{\partial t}\left[\int_0^r f_r(m,\phi,t,T)dm\right] \\
& \quad +\frac{\partial}{\partial t}g(\phi,t,T) + \frac{1}{2}\sigma^r(r,t)^2\frac{\partial}{\partial r}f_r(r,\phi,t,T) \qquad (36) \\
& \quad +(\sigma^f(r,t,T) - f_r(r,\phi,t,T)\sigma^r(r,t))\left(\frac{\mu^\phi(r,\phi,t)}{\sigma^\phi(r,\phi,t)} - \frac{1}{2}\frac{\partial \sigma^\phi(r,\phi,t)}{\partial \phi}\right) \\
\frac{\sigma^f(r,t,T) - f_r(r,\phi,t,T)\sigma^r(r,t)}{\sigma^\phi(r,\phi,t)} &= \frac{\partial}{\partial \phi}\left[\int_0^r f_r(m,\phi,t,T)dm\right] + \frac{\partial g(\phi,t,T)}{\partial \phi} \qquad (37) \\
\mu^r(r,\phi,t) &= -\frac{\partial}{\partial t}\left[\int_0^r f_r(m,\phi,t,T)dm\right]\Bigg|_{T=t} - \frac{\partial g(\phi,t,T)}{\partial t}\Bigg|_{T=t} \qquad (38)
\end{aligned}
$$

Proof: Suppose that $\sigma^f(r,t,T)$ is allowable in a general two-state Markovian paradigm. Then since the interest dynamics are given by (27-28), (30-35) hold. To obtain (36), substitute (32-34) into (30). (37) can be obtained from (33) and (35). Finally, to obtain (38), evaluate (36) at $T = t$.

Now suppose that (36-38) hold for some $\sigma^f(r,t,T)$, $f_r(r,\phi,t,T)$, $g(t,T,\phi)$, $\mu^\phi(r,\phi,t)$, $\mu^r(r,\phi,t)$ and $\sigma^\phi(r,\phi,t)$. Then define a two-state Markovian interest rate process by $dr = \mu^r(r,\phi,t)dt + \sigma^f(r,t,t)dz(t)$ and $d\phi = \mu^\phi(r,\phi,t)dt + \sigma^\phi(r,\phi,t)dz(t)$, with a corresponding forward rate volatility $\sigma^f(r,t,T)$. By (29-35), (36-38) can be obtained with the *same* $\sigma^f(r,t,T)$ that was orginally considered. Therefore the volatility stucture is indeed given by a two-state Markovian process. □

6 A Multi-factor Two-State Markovian Paradigm

To accurately value interest rate derivatives in many situations, a multifactor model is needed. The results for the one-factor Markovian models can be readily extended to two or factors. Under an n-factor HJM model, the forward rates follow

$$df(t,T) = \sum_{i=1}^{n} df_i(t,T) = \sum_{i=1}^{n} \sigma_i^f(\omega,t,T) \int_t^T \sigma_i^f(\omega,t,u)du\,dt + \sigma_i^f(\omega,t,T)dz_i(t) \tag{39}$$

where $\boldsymbol{\sigma}^f(\omega,t,T) = \{\sigma_i^f(\omega,t,T)\}$ is an adapted process on $\mathbf{R}^n$, and $\mathbf{z}(t) = \{z_i(t)\}$ is standard Brownian motion on $\mathbf{R}^n$, under an equivalent martingale (risk-neutral) measure Q that arises from Girsanov's Theorem.

The forward rate process for the n-factor model can be thought of as the sum of n independent single-factor forward rate processes. Likewise, since $r(t) = f(t,t)$, the n-factor interest rate is the sum of n independent one-factor interest rate processes:

$$r(t) = \sum_{i=1}^{n} r_i(t)$$

As in (2-3), the evolution of each pair of the n interest rate and state variable components is given by

$$dr_i = \mu_i^r(r_i,\phi_i,t)dt + \sigma_i^r(r_i,t)dz_i(t) \tag{40}$$
$$d\phi_i = \mu_i^\phi(r_i,\phi_i,t)dt \tag{41}$$

Since each of the pair of interest rate and state variable factor is independent of each other, we can consider the problem of finding a suitable volatility term, $\sigma_i^f(\omega,t,T)$, separately for each of the n factors.

Example 6.1 Consider a two-factor, two-state variable model given by

$$\sigma_1^f(r,t,T) = \sigma^r(r,t)e^{\left(-\int_t^T \kappa_1(\tau)\,d\tau\right)} \tag{42}$$
$$\sigma_2^f(r,t,T) = b(t)\cos\left(-\int_t^T \kappa_2(\tau)\,d\tau\right) \tag{43}$$

This example is important because it captures some of the important features of the general two-factor HJM model. Under the HJM approach, $\sigma_1^f(r,t,T)$ and $\sigma_2^f(r,t,T)$ functions are estimated from historical data using statistical techniques such as principal components analysis. It turns out that, with the appropriate choice of κ_1 and κ_2, $\sigma_1^f(r,t,T)$ and $\sigma_2^f(r,t,T)$ might be able to come close to matching the observed functions. So this example might almost be as accurate as using a two-factor HJM approach, but would be much faster, because it is Markovian with respect to two state variables.

Interest rate derivatives can be valued with model by using an extension of the lattice methods derived by Li, Ritchken and Sankarasubramanian (1995).

7 Conclusion

In this paper, necessary and sufficient conditions on the volatility structure of forward rates are identified for the term structure to follow a two-state Markov process. The class of volatility structures discussed here is a superset of the volatility structures considered in RS (1995).

It remains for future research to identify conditions for the general two-state Markov models to fit any initial term structure. This would involve extending Theorem 3.2 to the case where the state variable has a diffusion term. It also remains for future research to extend the lattice methods in Li, Ritchken and Sankarasubramanian (1995) to general multifactor models and to conduct empirical tests comparing model considered in this paper to the HJM approach.

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