Does Observed Skewness and Kurtosis Imply a Stochastic Volatility or Jump Diffusion Model? *

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Abstract

This paper considers two classes of volatility models, stochastic volatility and jump-diffusion models. It calculates the term structure of skewness and kurtosis of the distributions given by these models and compares that to the skewness and kurtosis as implied by empirical data.

1 Introduction

It is well known that empirical prices of options on equities differ from the Black-Scholes model. On average, Black-Scholes tends to overprice options that are at the money, and underprice options that are significantly in or out of the money. In particular, it tends to significantly underprice options with lower strike levels (i.e. out-of-the-money puts and in-the-money calls.)

In fact, since the crash of 1987, when compared with the normal return distribution under Black-Scholes, implied index distributions have had fat tails (positive kurtosis), especially at lower strike levels (negative skewness).

In order to fit the observed option prices, we consider extensions of the Black-Scholes model. Stochastic volatility and jump-diffusion are two of extensions of Black-Scholes that are studied in this paper. Stochastic volatility assumes that stock prices do follow a Geometric Brownian motion but relaxes the assumption that this stochastic process has a constant volatility. On the other hand, jump-diffusion augments the Geometric Brownian Motion with a Poisson jump process.

The formulas for the first four moments of the distribution are derived for a wide class of jump-diffusion and stochastic volatility models. When no analytical solution is available, a numerical method is used to compute the moments.

We use these moments to compute the term structure of skewness and kurtosis. We then compare this against the observed term structures of skewness and kurtosis. To extract the observed skewness and kurtosis we first extract the local volatility surface (Derman, Kani, and Zou 1996) by using the Edgeworth expansion (Cramer 1946). We then run Monte Carlo simulations over the local volatility surface to obtain the distributions implied by the option prices.

2 Jump-Diffusion

Jump-diffusion models extend the Black-Scholes model by adding a Poisson "jump" process to the return distribution.

2.1 The Model

One of the main underlying assumptions of the Black-Scholes models is that stock index level, (S_t), follows a Geometric Brownian Motion as given by

$$S_t = S_0 e^{\mu t + \sigma z_t}$$

Therefore the cumulative return process, $R_t = \ln \frac{S_t}{S_0}$, follows

$$dR_t = \mu dt + \sigma dz_t$$

In this section, we augment this process with a Possion process, (N_t) that generates jumps. We assume that (R_t) is given by the stochastic process

$$dR_t = \mu dt + \sigma dz_t + G_t dN_t$$

Where G_t and $G_{t'}$ have a common distribution, G, and are independent of each other if $t \neq t'$.

We denote the moments of the distribution G by $v_n = E[x^n]$ where $x \sim G$. We assume that the first four moments are finite.

Finally, for $n = 0, 1, 2, ...,$,

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

2.2 Characteristic Function of the Return Process

Let $\xi(x, \tau; u)$ be the characteristic function of $R_{t+\tau}$ given that $R_t = x$. The characteristic function is defined as

$$\xi(x, \tau; u) = E_t[e^{iuR_{t+\tau}}] = \int_{-\infty}^{\infty} e^{iuw} dF(w)$$

where u is a real variable, and $F(w)$ is the distribution function of $R_{t+\tau}$ given that $R_t = x$.

2.3 Kolmogorov Backward Equation

We now derive a Kolmogorov Backward PDE. Note that

$$\xi(x, \tau; u) = E_t[e^{iuR_{t+\tau}}] = E_t\left[E_{t+\Delta t}[e^{iuR_{t+\tau}}]\right] = E_t[\xi(x', \tau - \Delta t; u)]$$

where $R_{t+\Delta t} = x'$. Now,

$$\begin{aligned}
E_t[\xi(x', \tau - \Delta t; u)] &= (1 - \lambda\Delta t)E_t[\xi(x', \tau - \Delta t; u)| \text{ no jump }] \\
&\quad + \lambda\Delta t E_t[\xi(x', \tau - \Delta t; u)| \text{ jump }] + o(\Delta t) \\
&= (1 - \lambda\Delta t)E_t\left[\xi(x, \tau; u) + (x' - x)\frac{\partial\xi(x, \tau; u)}{\partial x}\right. \\
&\quad \left. + \frac{1}{2}(x' - x)^2\frac{\partial^2\xi(x, \tau; u)}{\partial x^2} - \Delta t\frac{\partial\xi(x, \tau; u)}{\partial\tau}\right] \\
&\quad + \lambda\Delta t E_t\left[\xi(x + G, \tau, u)\right] + o(\Delta t) \\
&= \xi(x, \tau; u) + \Delta t\left(\mu\frac{\partial\xi(x, \tau; u)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2\xi(x, \tau; u)}{\partial x^2}\right. \\
&\quad \left. - \frac{\partial\xi(x, \tau; u)}{\partial\tau} + \lambda E_t\left[\xi(x + G, \tau, u) - \xi(x, \tau, u)\right]\right) + o(\Delta t)
\end{aligned}$$

Since $\xi(x, \tau; u) = E_t[\xi(x', \tau - \Delta t; u)]$, the differential equation for $\xi(x, \tau, u)$ is given by

$$\mu\frac{\partial\xi(x, \tau; u)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2\xi(x, \tau; u)}{\partial x^2} - \frac{\partial\xi(x, \tau; u)}{\partial\tau} + \lambda E_t\left[\xi(x + G, \tau, u) - \xi(x, \tau, u)\right] = 0$$

With the initial condition

$$\xi(x, 0; u) = E_t[e^{iuR_t}] = e^{iux}$$

2.4 PDE Solution

The solution to this differential equation is

$$\xi(x,\tau,u) = e^{iux+i\mu\tau u-\frac{1}{2}\sigma^2\tau u^2+\lambda\tau E_t\left[e^{iuG}-1\right]}$$

Denote the returns over a period of length τ by $r_t(\tau) = R_{t+\tau} - R_t$. Now we define the characteristic function of $r_t(\tau)$,

$$\zeta(\tau,u) = E_t\left[e^{iur_t(\tau)}\right] = e^{-iuR_t}E_t\left[e^{iuR_{t+\tau}}\right] = e^{-iux}\xi(x,\tau,u) = e^{i\mu\tau u-\frac{1}{2}\sigma^2\tau u^2+\lambda\tau E_t\left[e^{iuG}-1\right]}$$

Since $r_t(\tau)$ is independent of t, we can simply denote it by $r(\tau)$.

3 Skewness and Kurtosis of Jump-Diffusion Returns

Intuitively, the skewness measures the difference between the upper and lower tail of the return distribution. For example, an index with large negative skewness has a higher probability of an abnormally large negative return than an abnormally large positive return. The kurtosis measures the relative size of the tails, as compared to the tails of a normal return distribution. (Under the normal returns in Black-Scholes, kurtosis is 3, and the skewness is 0.) For example, an index with a kurtosis greater than 3 has a higher probability of abnormally large price movements, as compared to an index with normally distributed returns.

We now calculate the skewness and kurtosis for returns under jump-diffusion models.

3.1 Moments of the Return Distribution

Let m_n be the n-th moment of the return $r(\tau)$ and let $f(w)$ be the distribution function of $r(\tau)$. Then

$$\begin{aligned} m_n &= E_t[r_t(\tau)^n] = \int_{-\infty}^{\infty} w^n df(w) = \frac{1}{i^n}\int_{-\infty}^{\infty} i^n w^n e^{iuw}|_{u=0} df(w) = \frac{1}{i^n}\int_{-\infty}^{\infty} \left.\frac{\partial^n e^{iuw}}{\partial u^n}\right|_{u=0} df(w) \\ &= \left.\frac{1}{i^n}\frac{\partial^n \zeta(\tau,u)}{\partial u^n}\right|_{u=0} \end{aligned}$$

The first moment, m_1, is given by

$$\begin{aligned} m_1 = \left.\frac{1}{i}\frac{\partial \zeta(\tau,u)}{\partial u}\right|_{u=0} &= \left.\frac{1}{i}\frac{\partial}{\partial u}\left[e^{i\mu\tau u-\frac{1}{2}\sigma^2\tau u^2+\lambda\tau\left[\int_{-\infty}^{\infty} e^{iuw}df(w)-1\right]}\right]\right|_{u=0} \\ &= \left.\frac{1}{i}A'e^{A}\right|_{u=0} \\ &= \mu\tau + \lambda\tau\int_{-\infty}^{\infty} w df(w) \\ &= \tau(\mu+\lambda v_1) \end{aligned}$$

Where $A = i\mu\tau u - \frac{1}{2}\sigma^2\tau u^2 + \lambda\tau\left[\int_{-\infty}^{\infty} e^{iuw} df(w) - 1\right]$. The second moment, m_2, is given by

$$\begin{aligned} m_2 = \frac{1}{i^2}\frac{\partial^2\zeta(\tau,u)}{\partial u^2}\bigg|_{u=0} &= \frac{1}{i^2}\frac{\partial^2}{\partial u^2}\left[e^{i\mu\tau u - \frac{1}{2}\sigma^2\tau u^2 + \lambda\tau\left[\int_{-\infty}^{\infty} e^{iuw} df(w) - 1\right]}\right]\bigg|_{u=0} \\ &= \frac{1}{i^2}(A'' + A'^2)e^A\bigg|_{u=0} \end{aligned}$$

The third moment, m_3 is given by

$$\begin{aligned} m_3 = \frac{1}{i^3}\frac{\partial^3\zeta(\tau,u)}{\partial u^3}\bigg|_{u=0} &= \frac{1}{i^3}\frac{\partial^3}{\partial u^3}\left[e^{i\mu\tau u - \frac{1}{2}\sigma^2\tau u^2 + \lambda\tau\left[\int_{-\infty}^{\infty} e^{iuw} df(w) - 1\right]}\right]\bigg|_{u=0} \\ &= \frac{1}{i^3}(A''' + 3A''A' + A'^3)e^A\bigg|_{u=0} \end{aligned}$$

The fourth moment, m_4 is given by

$$\begin{aligned} m_4 = \frac{1}{i^4}\frac{\partial^4\zeta(\tau,u)}{\partial u^4}\bigg|_{u=0} &= \frac{1}{i^4}\frac{\partial^4}{\partial u^4}\left[e^{i\mu\tau u - \frac{1}{2}\sigma^2\tau u^2 + \lambda\tau\left[\int_{-\infty}^{\infty} e^{iuw} df(w) - 1\right]}\right]\bigg|_{u=0} \\ &= \frac{1}{i^4}(A'''' + 4A'''A' + 3A''^2 + 6A''A'^2 + A'^4)e^A\bigg|_{u=0} \end{aligned}$$

3.2 Skewness and Kurtosis

The variance is given by

$$\begin{aligned} \text{Var}(r(\tau)) = E[(r_t(\tau) - E[r_t(\tau)])^2] = m_2 - m_1^2 &= \frac{1}{i^2}A''e^A\bigg|_{u=0} \\ &= \sigma^2\tau + \lambda\tau\int_{-\infty}^{\infty} w^2 df(w) \\ &= \tau(\sigma^2 + \lambda v_2) \end{aligned}$$

The skewness is given by

$$\begin{aligned} \frac{E[(r_t(\tau) - E[r_t(\tau)])^3]}{[\text{Var}(r(\tau))]^{\frac{3}{2}}} = \frac{m_3 - 3m_2m_1 + 2m_1^3}{(\tau(\sigma^2 + \lambda v_2))^{\frac{3}{2}}} &= \frac{1}{i^3(\tau(\sigma^2 + \lambda v_2))^{\frac{3}{2}}}A'''e^A\bigg|_{u=0} \\ &= \frac{\lambda\tau}{i^3(\tau(\sigma^2 + \lambda v_2))^{\frac{3}{2}}}\int_{-\infty}^{\infty} w^3 df(w) \\ &= \frac{\lambda v_3}{\sqrt{\tau}(\sigma + \lambda v_2)^{\frac{3}{2}}} \end{aligned}$$

The kurtosis is given by

$$\frac{E[(r_t(\tau) - E[r_t(\tau)])^4]}{[\text{Var}(r(\tau))]^2} = \frac{m_4 - 4m_3m_1 + 6m_2m_1^2 - 3m_1^4}{(\tau(\sigma^2 + \lambda v_2))^2} = \frac{(A'''' + 3A''^2)e^A\big|_{u=0}}{i^4(\tau(\sigma^2 + \lambda v_2))^2}$$

$$
\begin{aligned}
&= 3 + \frac{\lambda\tau}{(\tau(\sigma^2+\lambda v_2))^2}\int_{-\infty}^{\infty} w^4 df(w) \\
&= 3 + \frac{\lambda v_4}{\tau(\sigma^2+\lambda v_2)^2}
\end{aligned}
$$

4 Stochastic Volatility

Stochastic volatility models extend the Black-Scholes model by allowing the volatility to follow a stochastic process of its own.

4.1 The Model

Under the Black-Scholes model, the cumulative return of the index, R_t, follows a Brownian Motion as given by

$$dR_t = \mu dt + \sigma dz_t$$

In this section, we relax the assumption that σ is constant. Denote the volatility by $v_t = \sigma_t^2$. We assume that (R_t) and (v_t) are given by the stochastic processes

$$
\begin{aligned}
dR_t &= \mu dt + \sqrt{v}_t dz_t^{(1)} && (1) \\
dv_t &= a(\theta - v)dt + bv_t^{\alpha} dz_t^{(2)} && (2)
\end{aligned}
$$

where $dz_t^{(1)}$ and $dz_t^{(2)}$ are Wiener processes given with correlation ρ, that is

$$dz_t^{(1)} dz_t^{(2)} = \rho dt$$

In addition, we assume that v_t is positive. We can ensure this by placing a reflecting boundary at $v_t = 0$.

4.2 The Characteristic Function of the Return Process

Let $\xi(x, y, \tau; u)$ be the characteristic function of $R_{t+\tau}$ given that $R_t = x$ and $\sigma_t = y$. The characteristic function is defined as

$$\xi(x, y, \tau; u) = E_t[e^{iuR_{t+\tau}}] = \int_{-\infty}^{\infty} e^{iuw} dF(w)$$

where u is a real variable, and $F(x)$ is the distribution function of $R_{t+\tau}$ given that $R_t = x$ and $v_t = y$.

4.3 Kolmogorov Backward Equation

We now derive a PDE. Note that

$$\xi(x, y, \tau; u) = E_t[e^{iuR_{t+\tau}}] = E_t\left[E_{t+\Delta t}[e^{iuR_{t+\tau}}]\right] = E_t[\xi(x', y', \tau - \Delta t; u)]$$

where $R_{t+\Delta t} = x'$ and $v_{t+\Delta t} = y'$. Now, using Taylor Series,

$$\begin{aligned}
E_t[\xi(x', y', \tau - \Delta t; u)] &= E_t\left[\xi(x, y, \tau; u) + (x' - x)\frac{\partial \xi(x, y, \tau; u)}{\partial x}\right. \\
&\quad + \frac{1}{2}(x' - x)^2 \frac{\partial^2 \xi(x, y, \tau; u)}{\partial x^2} + (y' - y)\frac{\partial \xi(x, y, \tau; u)}{\partial y} \\
&\quad + \frac{1}{2}(y' - y)^2 \frac{\partial^2 \xi(x, y, \tau; u)}{\partial y^2} + (x' - x)(y' - y)\frac{\partial^2 \xi(x, y, \tau; u)}{\partial x \partial y} \\
&\quad \left. - \Delta t \frac{\partial \xi(x, y, \tau; u)}{\partial \tau}\right] + o(\Delta t) \\
&= \xi(x, y, \tau; u) + \Delta t \left(\mu \frac{\partial \xi(x, y, \tau; u)}{\partial x} + \frac{1}{2} y \frac{\partial^2 \xi(x, y, \tau; u)}{\partial x^2}\right. \\
&\quad + a(\theta - y)\frac{\partial \xi(x, y, \tau; u)}{\partial y} + \frac{1}{2} b^2 y^{2\alpha} \frac{\partial^2 \xi(x, y, \tau; u)}{\partial y^2} \\
&\quad \left. + b\rho y^{\frac{1}{2}+\alpha} \frac{\partial^2 \xi(x, y, \tau; u)}{\partial x \partial y} - \frac{\partial \xi(x, y, \tau; u)}{\partial \tau}\right) + o(\Delta t)
\end{aligned}$$

Since $\xi(x, y, \tau; u) = E_t[\xi(x', y', \tau - \Delta t; u)]$, the differential equation for $\xi(x, y, \tau, u)$ is given by

$$\begin{aligned}
&\mu \frac{\partial \xi(x, y, \tau; u)}{\partial x} + \frac{1}{2} y \frac{\partial^2 \xi(x, y, \tau; u)}{\partial x^2} + a(\theta - y)\frac{\partial \xi(x, y, \tau; u)}{\partial y} \\
&+ \frac{1}{2} b^2 y^{2\alpha} \frac{\partial^2 \xi(x, y, \tau; u)}{\partial y^2} + b\rho y^{\frac{1}{2}+\alpha} \frac{\partial^2 \xi(x, y, \tau; u)}{\partial x \partial y} - \frac{\partial \xi(x, y, \tau; u)}{\partial \tau} \quad = \quad 0
\end{aligned}$$

With the initial condition

$$\xi(x, y, 0; u) = E_t[e^{iuR_t}] = e^{iux}$$

4.4 PDE Solution

In general, there is no known analytical solution to this PDE. However in the case that $\alpha = \frac{1}{2}$, this equation is linear in y, and we "guess" the solution to be given by

$$\xi(x, y, \tau; u) = e^{iux + g(\tau; u) + yh(\tau; u)}$$

where

$$iu\mu - \frac{1}{2} y u^2 + a(\theta - y)\, h(\tau; u) + \frac{1}{2} b^2 y h(\tau; u)^2 + iub\rho y h(\tau; u) - \frac{dg(\tau; u)}{d\tau} - y\frac{dh(\tau; u)}{d\tau} \quad = \quad 0$$

This condition can be rewritten as

$$A(t; u) + B(t; u)y = 0$$

Therefore,

$$\begin{aligned} A(t;u) &= iu\mu + a\theta\, h(\tau;u) - \frac{dg(\tau;u)}{d\tau} = 0 && (3) \\ B(t;u) &= -\frac{1}{2}u^2 - a\, h(\tau;u) + \frac{1}{2}b^2 h(\tau;u)^2 + iub\rho h(\tau;u) - \frac{dh(\tau;u)}{d\tau} = 0 && (4) \end{aligned}$$

To solve (4), we can rewrite it as

$$\frac{2\, dh}{b^2h^2 + 2(iub\rho - a)h - u^2} = d\tau$$

Then integrating and applying the initial condition, $h(0;u) = 0$, gives

$$h(\tau;u) = \frac{-u^2(1-e^{-\gamma\tau})}{2\gamma + (\kappa-\gamma)(1-e^{-\gamma\tau})}$$

where

$$\begin{aligned} \gamma &= \sqrt{\kappa^2 + b^2u^2} \\ \kappa &= a - iub\rho \end{aligned}$$

Let $k(\tau;u)$ be denominator of $h(\tau,u)$. Then since $(\gamma-\kappa)(\gamma+\kappa) = b^2u^2$,

$$\begin{aligned} h(\tau;u) &= \frac{\kappa-\gamma}{b^2}\left(\frac{(\gamma+\kappa)(1-e^{-\gamma\tau})}{k(\tau;u)}\right) \\ &= \frac{\kappa-\gamma}{b^2}\left(\frac{k(\tau;u) - 2\gamma e^{-\gamma\tau}}{k(\tau;u)}\right) \\ &= \frac{\kappa-\gamma}{b^2}\left(1 - \frac{2k'(\tau;u)}{(\kappa-\gamma)k(\tau;u)}\right) \\ &= \frac{\kappa-\gamma}{b^2} - \frac{2k'(\tau;u)}{b^2k(\tau;u)} \end{aligned}$$

Then

$$\begin{aligned} g(\tau;u) &= \frac{a\theta}{b^2}\int_0^\tau \left[\kappa - \gamma + \frac{b^2 iu\mu}{a\theta} - \frac{2k'(s;u)}{k(s;u)}\right] ds \\ &= \frac{a\theta}{b^2}\left[\left(\kappa - \gamma + \frac{b^2 iu\mu}{a\theta}\right)\tau - 2\ln\frac{k(\tau;u)}{2\gamma}\right] \\ &= \frac{2a\theta}{b^2}\ln\left(\frac{2\gamma e^{\left(\kappa-\gamma+\frac{b^2 iu\mu}{a\theta}\right)\frac{\tau}{2}}}{k(\tau;u)}\right) \end{aligned}$$

Finally,

$$\zeta(y,\tau;u) = e^{-iux}\xi(x,y,\tau;u) = e^{g(\tau;u)+yh(\tau;u)}$$

where $\zeta(y,\tau;u)$ is defined (see Section 2.4) as the characteristic function of $r_t(\tau) = R_{t+\tau} - R_t$, the return over time τ.

5 Skewness and Kurtosis of Conditional Returns

We consider the skewness and kurtosis of $r_t(\tau)$, the return over time τ, conditional on the volatility being equal to y.

5.1 Moments of the Return Distribution

As in Section 3.1, the first four moments, m_1, m_2, m_3, m_4, are given by

$$
\begin{aligned}
m_1 = \frac{1}{i}\frac{\partial\zeta(y,\tau;u)}{\partial u}\bigg|_{u=0} &= \mu\tau \\
m_2 = \frac{1}{i^2}\frac{\partial^2\zeta(y,\tau;u)}{\partial u^2}\bigg|_{u=0} &= \frac{-\theta+\tau a\theta+\theta e^{-a\tau}+y-ye^{-a\tau}+\mu^2\tau^2 a}{a} \\
m_3 = \frac{1}{i^3}\frac{\partial^3\zeta(y,\tau;u)}{\partial u^3}\bigg|_{u=0} &= \frac{1}{a^2}[3b\rho\tau a\theta+3b\rho\theta\tau e^{-a\tau}a+3yb\rho-3yb\rho e^{-a\tau}+3a\theta\mu\tau e^{-a\tau}-3a\mu\tau ye^{-a\tau} \\
&\quad +3a\mu\tau y-3y\tau b\rho e^{-a\tau}a+\mu^3\tau^3a^2-6b\rho\theta+6b\rho\theta e^{-a\tau}+3\theta a^2\mu\tau^2-3a\theta\mu\tau] \\
m_4 = \frac{1}{i^4}\frac{\partial^4\zeta(y,\tau;u)}{\partial u^4}\bigg|_{u=0} &= \frac{1}{2a^3}[3\theta b^2e^{-2a\tau}+6a\theta^2e^{-2a\tau}+6ay^2e^{-2a\tau}-6yb^2e^{-2a\tau}-12a\theta e^{-2a\tau}y \\
&\quad +48\theta\mu\tau b\rho ae^{-a\tau}+48\theta b^2\tau\rho^2e^{-a\tau}a+72b^2\rho^2\theta e^{-a\tau}-48\theta\mu\tau b\rho a \\
&\quad +24\theta ba^2\mu\tau^2\rho+24b^2\rho^2\theta\tau a+12a^3\theta\mu^2\tau^3+2\mu^4\tau^4a^3+6\theta b^2\tau a \\
&\quad -72b^2\rho^2\theta+6a^3\theta^2\tau^2+12\theta b^2e^{-a\tau}-15\theta b^2-12a^2\theta^2\tau+6yb^2+6ay^2 \\
&\quad -12a\theta y-12a\theta^2e^{-a\tau}-12ay^2e^{-a\tau}+6a\theta^2+24yb^2\rho^2-12a^2\theta\mu^2\tau^2 \\
&\quad +12a^2\theta\tau y+12a^2\mu^2\tau^2y+24ba\rho\mu\tau y+12a^2\theta^2\tau e^{-a\tau}+24a\theta ye^{-a\tau} \\
&\quad -24yb^2\rho^2e^{-a\tau}-12y\tau^2b^2\rho^2e^{-a\tau}a^2-24y\tau b^2\rho^2e^{-a\tau}a-12y\tau b^2e^{-a\tau}a \\
&\quad +12\theta b^2\tau ae^{-a\tau}+12a^2\theta\mu^2\tau^2e^{-a\tau}-12a^2\mu^2\tau^2ye^{-a\tau}+12b^2\theta\tau^2\rho^2e^{-a\tau}a^2 \\
&\quad -12a^2\theta\tau ye^{-a\tau}-24ba\rho\mu\tau ye^{-a\tau}-24ba^2\rho\mu\tau^2ye^{-a\tau}+24ba^2\theta\rho\mu\tau^2e^{-a\tau}]
\end{aligned}
$$

5.2 Skewness and Kurtosis

The variance is given by

$$
\mathrm{Var}(r(\tau)) = E[(r_t(\tau)-E[r_t(\tau)])^2] = m_2-m_1^2 \quad = \quad \frac{-\theta+\tau a\theta+\theta e^{-a\tau}+y-ye^{-a\tau}}{a}
$$

The skewness is given by

$$
\frac{E[(r_t(\tau)-E[r_t(\tau)])^3]}{[\mathrm{Var}(r(\tau))]^{\frac{3}{2}}} = \frac{m_3-3m_2m_1+2m_1^3}{(\tau(\sigma^2+\lambda v_2))^{\frac{3}{2}}} =
$$

$$
3\frac{b\rho\left(\tau a\theta+\theta\tau e^{-a\tau}a+y-ye^{-a\tau}-y\tau e^{-a\tau}a-2\theta+2\theta e^{-a\tau}\right)}{\left(-\theta+\tau a\theta+\theta e^{-a\tau}+y-ye^{-a\tau}\right)^{3/2}\sqrt{a}}
$$

The kurtosis is given by

$$
\frac{E[(r_t(\tau)-E[r_t(\tau)])^4]}{[\mathrm{Var}(r(\tau))]^2} = \frac{m_4-4m_3m_1+6m_2m_1^2-3m_1^4}{(\tau(\sigma^2+\lambda v_2))^2} =
$$

$$\frac{3}{2a\left(-\theta+\tau a\theta+\theta e^{-a\tau}+y-ye^{-a\tau}\right)^2}[\theta b^2e^{-2a\tau}+2a\theta^2e^{-2a\tau}+2ay^2e^{-2a\tau}-2yb^2e^{-2a\tau}-4a\theta e^{-2a\tau}y+$$
$$16\theta b^2\tau\rho^2e^{-a\tau}a+24b^2\rho^2\theta e^{-a\tau}+8b^2\rho^2\theta\tau a+2\theta b^2\tau a-24b^2\rho^2\theta+2a^3\theta^2\tau^2+4\theta b^2e^{-a\tau}-5\theta b^2-$$
$$4a^2\theta^2\tau+2yb^2+2ay^2-4a\theta y-4a\theta^2e^{-a\tau}-4ay^2e^{-a\tau}+2a\theta^2+8yb^2\rho^2+4a^2\theta\tau y+4a^2\theta^2\tau e^{-a\tau}+$$
$$8a\theta ye^{-a\tau}-8yb^2\rho^2e^{-a\tau}-4y\tau^2b^2\rho^2e^{-a\tau}a^2-8y\tau b^2\rho^2e^{-a\tau}a-4y\tau b^2e^{-a\tau}a+$$
$$4\theta b^2\tau ae^{-a\tau}+4b^2\theta\tau^2\rho^2e^{-a\tau}a^2-4a^2\theta\tau ye^{-a\tau}]$$

6 Skewness and Kurtosis of Unconditional Returns

Once again, we consider the skewness and kurtosis of $r_t(\tau)$, but without any information about the volatility. The stationary density of volatility that evolves by the square-root process (1) when $\alpha=\frac{1}{2}$ is given by

$$\pi(v)=\frac{1}{\Gamma(\omega\theta)}\omega^{\omega\theta}v^{\omega\theta-1}e^{-\omega v}$$

where $\omega=\frac{2a}{b^2}$ and $v=0$. The characteristic function of unconditional returns is given by

$$\begin{aligned}\varsigma(\tau;u) &= E_t[e^{iur(\tau)}]=\int_0^\infty E_t[e^{iur(\tau)}|v_t=y]\pi(y)dy=\int_0^\infty\zeta(y,\tau;u)\pi(y)dy\\ &= e^{g(\tau;u)}\int_0^\infty e^{yh(\tau;u)}\pi(y)dy=e^{g(\tau;u)}\left(\frac{\omega}{\omega-h(\tau;u)}\right)^{\omega\theta}\end{aligned}$$

6.1 Moments of the Return Distribution

The first four moments, m_1,m_2,m_3,m_4, are given by

$$\begin{aligned}m_1=\frac{1}{i}\frac{\partial\varsigma(y,\tau;u)}{\partial u}\bigg|_{u=0} &= \mu\tau\\ m_2=\frac{1}{i^2}\frac{\partial^2\varsigma(y,\tau;u)}{\partial u^2}\bigg|_{u=0} &= \tau\left(\theta+\mu^2\tau\right)\\ m_3=\frac{1}{i^3}\frac{\partial^3\varsigma(y,\tau;u)}{\partial u^3}\bigg|_{u=0} &= \frac{-3b\rho\theta+\mu^3\tau^3a^2+3\theta a^2\mu\tau^2+3b\rho\tau a\theta+3b\rho\theta e^{-a\tau}}{a^2}\\ m_4=\frac{1}{i^4}\frac{\partial^4\varsigma(y,\tau;u)}{\partial u^4}\bigg|_{u=0} &= \frac{1}{a^3}[-24b^2\rho^2\theta+12b^2\rho^2\theta\tau a+6a^3\theta\mu^2\tau^3-3\theta b^2+\mu^4\tau^4a^3+\\ &\quad 12\theta b^2\tau\rho^2e^{-a\tau}a-12\theta\mu\tau b\rho a+12\theta\mu\tau b\rho ae^{-a\tau}+12\theta ba^2\mu\tau^2\rho+\\ &\quad 3\theta b^2\tau a+3a^3\theta^2\tau^2+24b^2\rho^2\theta e^{-a\tau}+3\theta b^2e^{-a\tau}]\end{aligned}$$

6.2 Skewness and Kurtosis

The variance is given by

$$\tau\theta$$

The skewness is given by

$$3\frac{b\rho\left(-1+a\tau+e^{-a\tau}\right)}{a^2\tau\sqrt{\tau\theta}}$$

The kurtosis is given by

$$3\frac{-8b^2\rho^2 + 4b^2\rho^2\tau a - b^2 + 4\tau b^2\rho^2 e^{-a\tau}a + b^2\tau a + a^3\theta\tau^2 + 8b^2\rho^2 e^{-a\tau} + b^2 e^{-a\tau}}{a^3\theta\tau^2}$$

7 Extracting an Implied Distribution from Option Prices

In this section, we obtain a return distribution that approximates a given series of option prices. The option prices are given by $C(K_i) = C(K_i, T)$ $(i = 1, 2, 3 \dots M)$ where K_i is the range of strike levels for options maturing at time T. We then seek an risk neutral implied probably distribution for the index level that is consistent with the series of option prices. Let $f(S) = f(S, T)$ be this implied probability density at index level S and time T conditioned on the index level being S_t at time t. Let $g(S) = g(S, T)$ be a log-normal distribution with volatility σ.

7.1 Edgeworth Expansion

We can approximate $f(S)$ in terms of $g(S)$ by using the Edgeworth expansion[1]. The characteristic functions of these density functions are given by their Fourier transforms.

$$\begin{aligned}
\hat{f}(u) &= \int_{-\infty}^{\infty} f(S)e^{iSu}dS = \sum_{n=0}^{\infty}\frac{(iu)^n}{n!}\mu_n \\
\hat{g}(u) &= \int_{-\infty}^{\infty} g(S)e^{iSu}dS = \sum_{n=0}^{\infty}\frac{(iu)^n}{n!}\nu_n
\end{aligned}$$

where $\mu_n = \int S^n f(S)dS$ and $\nu_n = \int S^n g(S)dS$. Furthermore, the logarithms of the characteristic functions are given by

$$\begin{aligned}
\log\hat{f}(u) &= \log\left(1 + \sum_{n=1}^{\infty}\frac{(iu)^n}{n!}\mu_n\right) = \sum_{n=1}^{\infty}\frac{(iu)^n}{n!}\chi_n \\
\log\hat{g}(u) &= \log\left(1 + \sum_{n=1}^{\infty}\frac{(iu)^n}{n!}\nu_n\right) = \sum_{n=1}^{\infty}\frac{(iu)^n}{n!}\kappa_n
\end{aligned}$$

χ_n and κ_n are called the cumulants of the distributions $f(S)$ and $g(S)$, respectively. In particular we have

$$\begin{aligned}
\chi_1 &= \mu_1 \\
\chi_2 &= \mu_2 - \mu_1^2 \\
\chi_3 &= \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3
\end{aligned}$$

Here we used the identity that

$$\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

Now,

$$\log\left[\frac{\hat{f}(u)}{\hat{g}(u)}\right] = \sum_{n=1}^{\infty}\frac{(iu)^n}{n!}(\chi_n - \kappa_n)$$

[1] See Cramer 1946, p. 228

Then,

$$\frac{\hat{f}(u)}{\hat{g}(u)} = e^{\sum_{n=1}^{\infty} \frac{(iu)^n}{n!}(\chi_n - \kappa_n)} = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} a_n$$

and

$$\hat{f}(u) = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} a_n \hat{g}(u)$$

where a_n is given by

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= \chi_1 - \kappa_1 = \mu_1 - \nu_1 = 0 \\
a_2 &= \chi_2 - \kappa_2 + a_1^2 \\
a_3 &= \chi_3 - \kappa_3 + 3a_1(\chi_2 - \kappa_2) + a_1^3 \\
&\vdots
\end{aligned}$$

We now write the inverse Fourier transform of the characteristic function to obtain

$$\begin{aligned}
f(S) &= \int_{-\infty}^{\infty} \hat{f}(u) e^{-iSu} du \\
&= \sum_{n=0}^{\infty} \int \frac{(iu)^n a_n}{n!} \hat{g}(u) e^{-iSu} du \\
&= g(S) + \sum_{n=2}^{\infty} \frac{(-1)^n a_n}{n!} \frac{\partial^n g(S)}{\partial S^n}
\end{aligned}$$

This is known as the Edgeworth expansion.

7.2 Fitting Observed Option Prices

We now write the call prices

$$\begin{aligned}
C(K_i) &= e^{-r(T-t)} \int_{K_i}^{\infty} (S - K_i) f(S) dS \\
&= e^{-r(T-t)} \int_{K_i}^{\infty} (S - K_i) g(S) dS + e^{-r(T-t)} \sum_{n=2}^{\infty} \int_{K_i}^{\infty} (S - K_i) \frac{(-1)^n a_n}{n!} \frac{\partial^n g(S)}{\partial S^n} dS \\
&= C_{\mathrm{BS}}(K_i) + e^{-r(T-t)} \sum_{n=2}^{\infty} \frac{(-1)^n a_n}{n!} \left. \frac{\partial^{(n-2)} g(S)}{\partial S^{(n-2)}} \right|_{S=K_i}
\end{aligned}$$

Where $C_{\mathrm{BS}}(K_i) = C_{\mathrm{BS}}(K_i, S, T, \sigma)$ is the Black-Scholes call price with volatility σ. To obtain the last equation, we used integration by parts. In practice, we truncate the expansion at the Nth order:

$$C(K_i) \approx C_{\mathrm{BS}}(K_i) + e^{-r(T-t)} \sum_{n=2}^{N} \frac{(-1)^n a_n}{n!} \left. \frac{\partial^{(n-2)} g(S)}{\partial S^{(n-2)}} \right|_{S=K_i}$$

We now wish to seek the σ and $\{a_n\}$ that best fit the observed option prices. We define an objective function:

$$\Omega(\sigma, a_2, ..., a_N) = \sum_{i=1}^{M} \left[C(K_i) - C_{\text{obs}}(K_i)\right]^2$$

where C_{obs} is the observed market prices for call options. In practice, out-of-the-money calls and puts are used, since these options are more liquid. To value in-the-money calls, put-call parity is used with the values of the out-of-the-money puts.

8 Observed Skewness and Kurtosis

8.1 Options Data

We used daily options data on the S&P 500 index from settlement dates November 23, 1995 to August 14, 1998. The option prices were quoted in terms of volatility at different strike levels and expiration dates. Typically there were about ten to twenty strike levels and expirations on the Saturday after third Friday of every month. We had expirations up to two years, although the data was sparse for expirations after one year.

For each set of options with a common settlement and maturity date, we considered the Black-Scholes implied volatility of options at different strike levels. The implied volatility typically had a "smile" or "smirk" whereas the implied volatilities for deep out-of-the-money and deep in-the-money options was significantly larger than for at-the-money options. This effect was especially pronounced for out-of-the-money puts and in-the-money calls. (That is, for strikes below the forward price of the index.) This differed from the implied option volatilities given by the Black-Scholes model, where the volatility was constant for every strike.

8.2 Extracting the Implied Distribution from Option Prices

Let the settlement and the maturity date be fixed. Denote the implied option volatilities by $\Sigma(K_i)$ $(i = 1, 2, 3 \ldots M)$ where K_i is the range of strike levels for options with the given settlement and expiration date. Here we use the implied volatilities of out-of-the-money calls and puts.[2] Then we compute the call prices by the Black-Scholes formula for the value of a call option, $C(K_i) = C_{\text{BS}}(S, t, K_i, T, \Sigma(K_i))$.

We then use the Edgeworth expansion outlined in the previous section to approximate a distribution function, $f(S)$, that would give rise to the call option prices $C(K_i)$.

8.3 Extracting the Local Volatility Surface from the Implied Distribution

Using the Edgeworth expansion, we can obtain an approximation to the actual distribution of the S&P 500 index that fits the observed option prices. We wish to find the moments of this distribution. However, the area under this distribution does not necessarily add to one, and the distribution might oscillate for values S far away from the forward price.

[2] In theory, the implied volatilities for a call and a put option at the same strike should be the same, according to Put-Call parity. However, since markets are not perfectly frictionless and liquid, there exist discrepancies between the implied volatilities of call and put options. In particular, since out-of-the-money options tend to be more heavily traded and more liquid than in-the-money calls, we use the volatilities of the out-of-the-money options, since they are more accurate.

To construct a more smooth and robust distribution, we first calculate the local volatility surface from the implied distribution, and then we run Monte Carlo simulations over this surface. The resulting terminal values of the Monte Carlo simulation give rise to a robust distribution and its corresponding moments.

Under a local volatility model, the index evolves according to the following stochastic process

$$d \ln S = r dt + \sigma(S,t) dz_t$$

where $\sigma(S,t)$ is the local volatility function which depends on both the index level S and the time t.

In order to extract the local volatility surface from the implied volatilities, we will now express $\sigma(S,t)$ as a function of the derivatives of call option prices, which we in turn express as the function of the implied distribution.

Let $p(S,t,S',T)$ be the transition probability density of the index process S. The Fokker-Planck equation[3] gives

$$\frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\sigma^2(S',T)S'^2 p\right) = \frac{\partial p}{\partial T} + r\frac{\partial}{\partial S'}\left(S'p\right)$$

The call price is given by

$$C = C(S,t,K,T) = e^{-r(T-t)}\int_K^\infty (S'-K)\, p\, dS'$$

Now

$$\begin{aligned}
\frac{\partial C}{\partial T} &= e^{-r(T-t)}\int_K^\infty (S'-K)\left[\frac{\partial p}{\partial T} - rp\right] dS' \\
&= e^{-r(T-t)}\int_K^\infty (S'-K)\left[\frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\sigma^2(S',T)S'^2 p\right) - r\frac{\partial}{\partial S'}\left(S'p\right) - rp\right] dS' \\
&= e^{-r(T-t)}\int_K^\infty \left[-\frac{1}{2}\frac{\partial}{\partial S'}\left(\sigma^2(S',T)S'^2 p\right) + S'rp - (S'-K)rp\right] dS' \\
&= e^{-r(T-t)}\left[\frac{1}{2}\sigma^2(S',T)S'^2 p + \int_K^\infty KrpdS'\right] \\
&= \frac{1}{2}\sigma^2(K,T)K^2\frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K}
\end{aligned}$$

We have arrived at Dupire's equation (1994)

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2(K,T)K^2\frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K}$$

We can now write the local volatility in terms of the derivatives of the call price

$$\sigma^2(K,T) = 2\frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{K^2\frac{\partial^2 C}{\partial K^2}}$$

[3]See Gihman, 1972, p. 102-4

where

$$\begin{aligned}
\frac{\partial C}{\partial K} &= -e^{-r(T-t)} \int_K^\infty f(S)dS \\
&= -e^{-r(T-t)} N(d_2) + e^{-r(T-t)} \sum_{n=2}^{N} \frac{(-1)^n a_n}{n!} \left. \frac{\partial^{(n-1)} g(S)}{\partial S^{(n-1)}} \right|_{S=K} \\
\frac{\partial^2 C}{\partial K^2} &= -e^{-r(T-t)} f(K)
\end{aligned}$$

where $N(x)$ is the cumulative probability distribution of the standard normal distribution and

$$d_2 = \frac{\ln \frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

To calculate $\frac{\partial C}{\partial T}$, we estimate the prices of options with strikes K_i $(i = 1, 2, 3 \ldots M)$ maturing at $T + \Delta T$, and calculate the implied distribution and call prices.

8.4 Using Monte Carlo Simulation to calculate skewness and kurtosis

We estimate the implied distribution by running a Monte Carlo simulation over the local volatility surface calculated in the previous section.

$$\Delta S_i = (r - \frac{1}{2}\sigma(S_i, t_i)^2) S_i \Delta t + \sigma(S_i, t_i) S_i \Delta z_i \sqrt{(\Delta t)}$$

We then calculate the first four moments of the resulting distribution at time T as well as the skewness and kurtosis.

9 Comparisons to Observed data

9.1 Jump-Diffusion

A variety of jump-diffusion parameters was tried, and none fit the observed term structure for skewness nor kurtosis. This is because the term structure for both of these statistics is a monotone decreasing function of τ, proportional to $\frac{1}{\sqrt{\tau}}$ in skewness and $\frac{1}{\tau}$ in kurtosis, while the observed term structures are humped.

Figure 1 shows the term structure for a jump-diffusion process, with an average of five jumps a year. Each of these jumps is distributed with a mean of -4.5% and a standard deviation of 7%. Although these values are rather high, this model does not generate enough skewness and kurtosis, especially for large expiration times.

9.2 Stochastic Volatility

Unlike jump-diffusion, stochastic volatility matches the observed term structure quite well. Both the observed term structure and the term structure generated by the stochastic volatility model are humped.

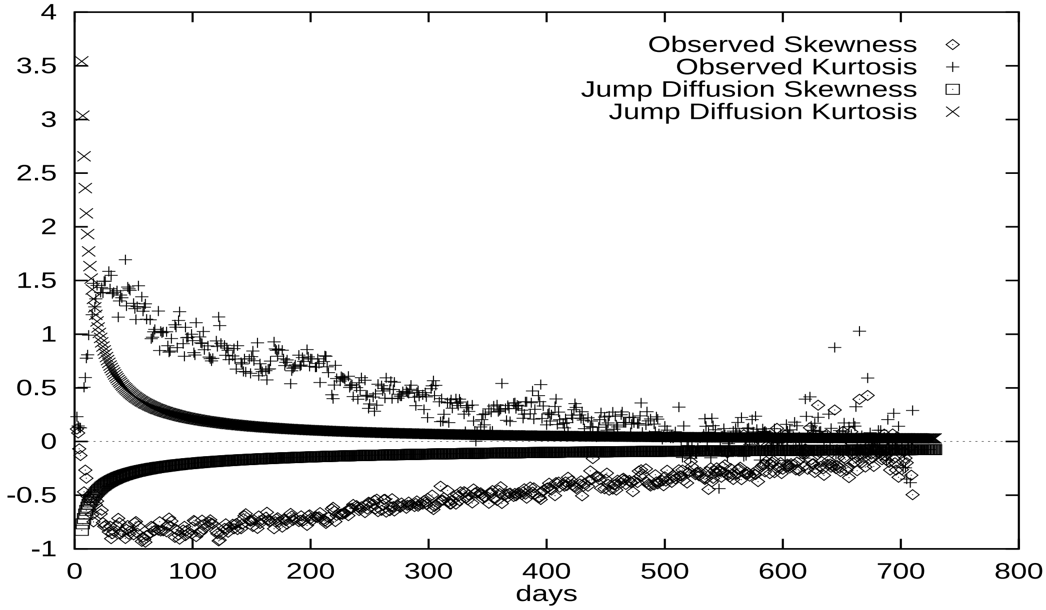

Figure 1: Skewness and Kurtosis Term Structures under Jump Diffusion process with $\lambda = 5, \mu = -0.045, \gamma = 0.07, \sigma = 0.116$

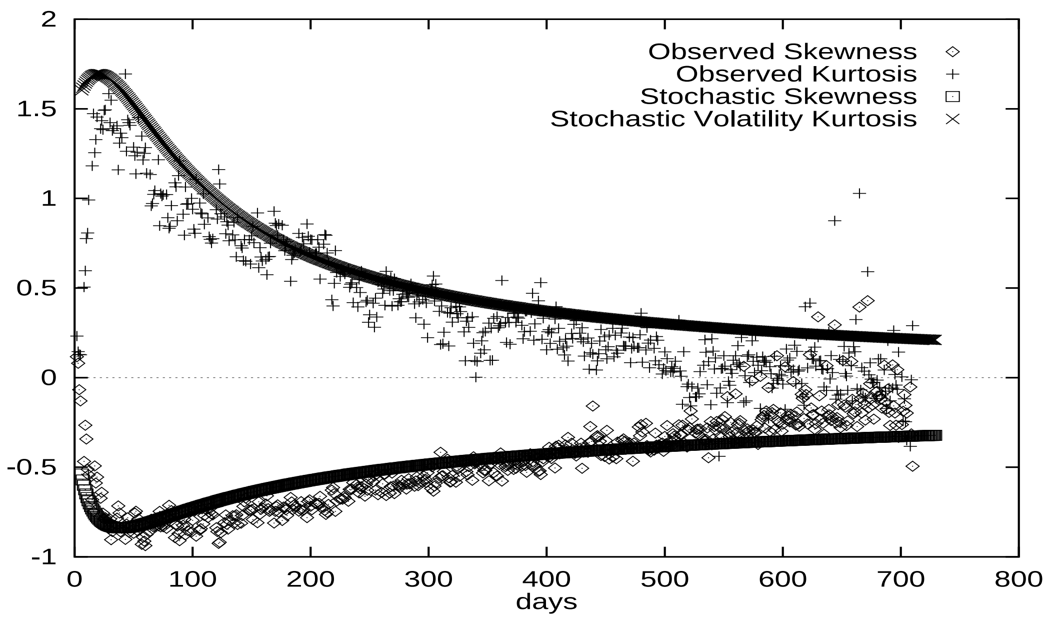

Figure 2: Skewness and Kurtosis Term Structures under Stochastic Volatility process with $\mu = 0.06, a = 20, \theta = 0.03, b = 0.77, \alpha = 0.5, \rho = -0.7$

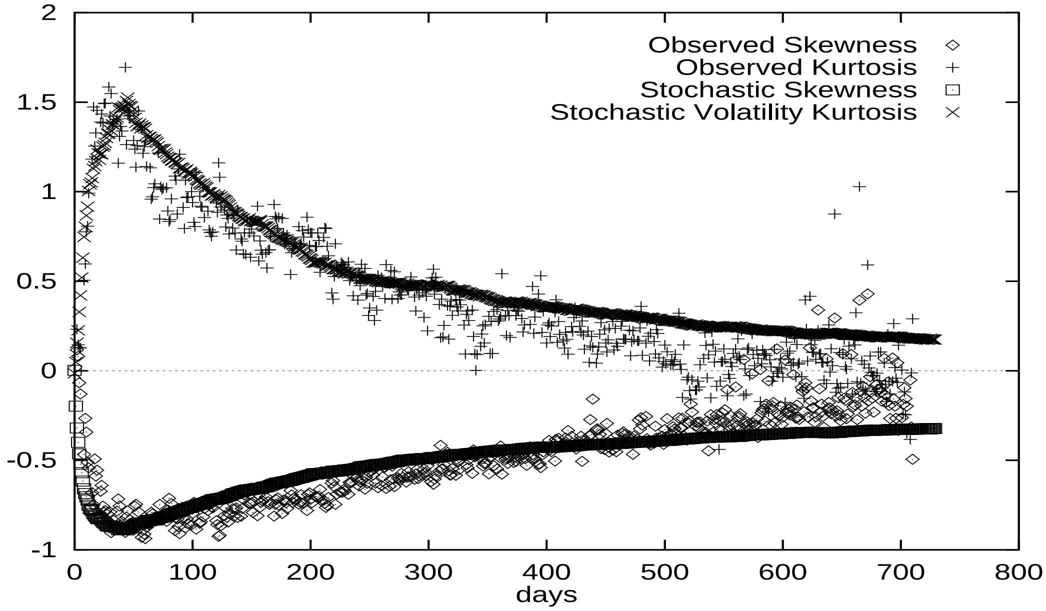

Figure 3: Skewness and Kurtosis Term Structures under Stochastic Volatility process with $\mu = 0.06, a = 20, \theta = 0.03, b = 0.85, \alpha = 0.3, \rho = -0.7$

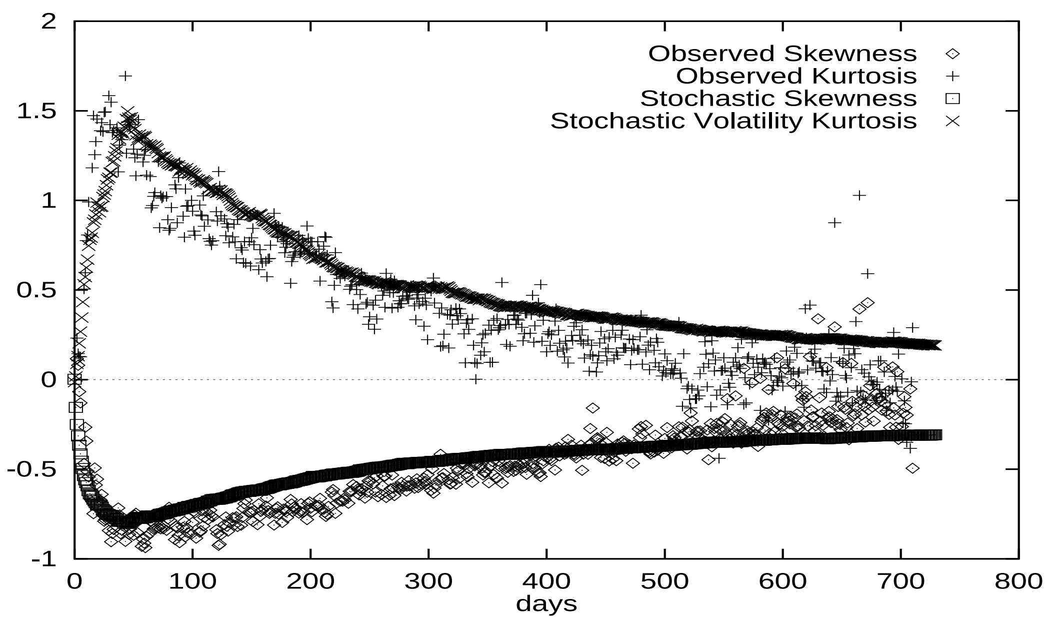

Figure 4: Skewness and Kurtosis Term Structures under Stochastic Volatility process with $\mu = 0.06, a = 20, \theta = 0.03, b = 0.59, \alpha = 0.8, \rho = -0.81$

Figure 2 shows the term structure for a stochastic volatility process with $\alpha = 0.5$, the only value at which we have a analytical solution. At other values of α, we have to estimate the skewness and kurtosis by using Monte Carlo simulations. Figures 3 and 4 show the result of estimating the term structures for $\alpha = 0.3$ and $\alpha = 0.8$, respectively. In these examples, 100,000 simulations were run.

In the Figure 2 ($\alpha = 0.5$), ρ, the correlation between the return and the volatility is -0.7, which is close to the observed correlation of -0.58 for the S&P 500 (Mather and Zou 1998). However, a, the volatility mean-reversion factor is much higher than the mean-reversion of observed implied volatilities. A plausible explanation for this is that the volatility process could be composed as the sum of two processes, a low frequency process that has a low rate of mean reversion, and a high frequency process that has a high rate of mean reversion. Since the high frequency process would die out quickly, it would not be reflected in the implied volatilities, but it would affect the skewness and kurtosis.

10 Conclusion

In the first part of the paper, we derived closed form solutions for the skewness and kurtosis of return distributions for jump-diffusion and stochastic volatility models, two widely studied extensions of the Black-Scholes model. In the second part of this paper, we developed numerical methods for extracting skewness and kurtosis from observed option prices. Finally, we compared the skewness and kurtosis generated by these two models to the skewness and kurtosis extracted from empirical data. We found that skewness and kurtosis term structures generated by stochastic volatility matched the observed term structures much more closely than the term structures generated by the jump-diffusion models.

Appendix A

This code calculates the formulas for skewness and kurtosis for both jump diffusion and stochastic volatility for $\alpha = \frac{1}{2}$. It also estimates the skewness and kurtosis for stochastic volatility for any value of α by using Monte Carlo simulation.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <assert.h>
#include "nrutil.h"

#define MONTE_CARLO_SIM 10000
#define SIGMA 0.20

///////////////////////////////////////////
// Jump Diffusion parameters

double sigma;
double lambda=5;
double v2,v3,v4;
double jumpmu=-0.045,jumpgamma=0.07;

///////////////////////////////////////////
// Stochastic Volatility parameters

double mu=0.06;
double a=20;
double theta=0.03;
double b=0.85;
double alpha=0;
double rho=-0.7;

double t,T=2.0;
double deltat=1.0/365.0;
double r,v;

double jumpDiffusionSkew(double tau){
    v2=jumpmu*jumpmu+jumpgamma*jumpgamma;
    v3=jumpmu*jumpmu*jumpmu+3*jumpmu*jumpgamma*jumpgamma;
    sigma=sqrt(SIGMA*SIGMA - lambda*v2);
    return lambda*v3/(sqrt(tau)*pow(sigma+lambda*v2,1.5));
}

double jumpDiffusionKurtosis(double tau){
    v2=jumpmu*jumpmu+jumpgamma*jumpgamma;
```

```
    v4=pow(jumpmu,4)+6*jumpmu*jumpmu*jumpgamma*jumpgamma+3*pow(jumpgamma,4);
    sigma=sqrt(SIGMA*SIGMA - lambda*v2);
    return lambda*v4/(tau*pow(sigma+lambda*v2,2.0));
}

double stochasticVolatilitySkew(double tau){
    return 3.0*b*rho*(-1.0+a*tau+exp(-a*tau))/(a*a)/tau/sqrt(tau*theta);
}

double stochasticVolatilityKurtosis(double tau){
    return 3.0/theta*(-8.0*b*b*rho*rho+4.0*b*b*rho*rho*tau*a-b*b+
           4.0*tau*b*b*rho*rho*exp(-a*tau)*a+b*b*tau*a+a*a*a*theta*tau*tau+
           8.0*b*b*rho*rho*exp(-a*tau)+b*b*exp(-a*tau))/(a*a*a)/(tau*tau)-3.0;
}

void main()
{
    long i,j,d,idum=-1;
    float dz1, dz2;
    double m1[1000], m2[1000], m3[1000], m4[1000], var, skew, kurtosis,tau,y;
    b=b*pow(theta,0.5-alpha);
    for(j=0;j<1000;j++)
        m1[j]=m2[j]=m3[j]=m4[j]=0;
    v=theta;
    fprintf(stderr,"b=%f\n",b);
    for(i=0;i<MONTE_CARLO_SIM;i++){
        if(i%1000==0)
            fprintf(stderr,"%d paths simulated\n",i);
        d=rand()%2+1;
        r=0;
        v=theta;
        for(j=t=0;t<T;t+=deltat){
            // dz1 & dz2 get random numbers with standard gaussian distribution
            // so that the correlation between them is rho
            dz1=gasdev(&idum);
            dz2=sqrt(1-rho*rho)*gasdev(&idum)+rho*dz1;

            // update r and v according to their SDE
            r+=mu*deltat + sqrt(v*deltat)*dz1;
            v+=a*(theta-v)*deltat + b * pow(v,alpha)*sqrt(deltat)*dz2;

            // reflecting boundary
            if(v<0){v=-v;}

            // update moments
```

```
            m1[j]+=r;
            m2[j]+=r*r;
            m3[j]+=r*r*r;
            m4[j]+=r*r*r*r;

            j++;
        }
    }
    for(j=0;j<1000;j++){
        m1[j]/=MONTE_CARLO_SIM;
        m2[j]/=MONTE_CARLO_SIM;
        m3[j]/=MONTE_CARLO_SIM;
        m4[j]/=MONTE_CARLO_SIM;
        var = m2[j]-m1[j]*m1[j];
        skew = (m3[j] - 3*m2[j]*m1[j] + 2*m1[j]*m1[j]*m1[j])/pow(var,1.5);
        kurtosis = (m4[j] - 4*m3[j]*m1[j] + 6*m2[j]*m1[j]*m1[j]
                - 3*m1[j]*m1[j]*m1[j]*m1[j])/(var*var)-3.0;

        tau=j*deltat;

        if(var>0){
            printf("%d %f %f %f %f %f %f\n",j,jumpDiffusionSkew(tau),
                jumpDiffusionKurtosis(tau),stochasticVolatilitySkew(tau),
                stochasticVolatilityKurtosis(tau),skew,kurtosis);
        }
    }
}
```

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